CALCULUS AND LINEAR ALGEBRA, 2015/2016

MIDTERM EXAM 12/16/2015

FIRST NAME: ______________________

FAMILY NAME: ____________________

MATRICOLA N.: ____________________

1. Compute the following limit

$$\lim_{x\to 0} \frac{x^3+2x+1}{2x^2+x-1} \tan\left(\frac{1}{x}\right)$$

Solution:

$$\lim_{x\to\infty} \frac{x^2+2x+1}{2x^2+x-1} \left(\frac{\dfrac{\sin\left(\dfrac{1}{x}\right)}{\dfrac{1}{x}}}{\cos\left(\dfrac{1}{x}\right)} \right) = \frac{1}{2}\cdot 1 = \frac{1}{2}.$$

2. For $t \in \mathbb{R}$, find the solution $y(t)$ of the Cauchy problem

$$\begin{cases} y' = \dfrac{1}{1+t^2} y \\ y(0) = \left(\dfrac{\pi}{2}\right)^{-\frac{1}{2}} \end{cases}.$$

Solution:
The solution satisfies

$$\int_{y(0)}^{y(t)} \frac{1}{s} ds = \int_0^t \frac{1}{s^2+1} ds$$

which implies

$$\log y(t) + \frac{1}{2}\log\left(\frac{\pi}{2}\right) = \arctan t$$

and then

$$y(t) = e^{\arctan t}\left(\frac{\pi}{2}\right)^{-\frac{1}{2}}.$$

3. Consider the function $f(x) = \dfrac{2x^2}{x^2+1}$, find its domain, sign, limits at the boundaries(asymptotes), maxima, minima, inflection points and sketch the graph.

Solution:

Domain: $\mathbb{R}$
Intersections with axis: $O = (0,0)$
Sign: $f(x) \geq 0 \quad \forall x \in \mathbb{R}$
Boundaries: $\lim_{x\to\pm\infty} f(x) = 2$
First Derivative: $f'(x) = \dfrac{4x(x^2+1) - 2x\cdot 2x^2}{(x^2+1)^2} = \dfrac{4x}{(x^2+1)^2}$
Global minimum: $x = 0$
Second Derivative: $f''(x) = \dfrac{4(x^2+1)^2 - 4x\cdot 4x(x^2+1)}{(x^2+1)^2} = \dfrac{4-12x^2}{x^2+1}$
Inflection points: $x = \pm\dfrac{1}{\sqrt{3}}$

4. Compute with the integration by part method the following definite integral

$$\int_0^{\pi} \sin x \cos x \, dx$$

Solution:

$$\int_0^{\pi} \sin x \cos x \, dx = \frac{1}{2} \int_0^{\pi} ((\sin x)^2)' dx = \frac{1}{2} \left[(\sin x)^2\right]_0^{\pi} = 0.$$

5. Using the inverse matrix method, solve the following linear system

$$\begin{cases} x + z = 0 \\ x + 2y + z = 2 \\ x - y = 1 \end{cases}$$

The linear system rewrites as $AX = b$ with

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & -1 & 0 \end{pmatrix} \quad , \quad b = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ .$$

The matrix A is invertible since $\det A = -2$. The inverse matrix is

$$A^{-1} = -\frac{1}{2} \begin{pmatrix} 1 & -1 & -2 \\ 1 & -1 & 0 \\ -3 & 1 & 2 \end{pmatrix} \ .$$

Therefore, multiplying by A^{-1} on both sides of the matrix equation, one gets

$$AX = b \iff A^{-1}AX = A^{-1}b \iff X = A^{-1}b \ .$$

In other words the system can be solved by computing $A^{-1}b$. Now:
Therefore the solution of the linear system is

$$x = 2 \ , \ \ y = 1 \ , z = -2 \ .$$

6. Find the eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$.

Solution:
A real number λ is an eigenvalue of A if and only if

$$\det(A - \lambda I) = 0 \iff \det \begin{pmatrix} -1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = 0 \iff$$
$$\iff \lambda^2 - 5 = 0 \ .$$

Therefore the eigenvalues of A are $\lambda = \sqrt{5}, -\sqrt{5}$.

A vector $\vec{v} \neq 0$ is an eigenvector of A associated to the eigenvalue λ if and only if $A\vec{v} = \lambda\vec{v}$, that is $(A - \lambda I)\vec{v} = 0$.
For the eigenvalue $\lambda = \sqrt{5}$, the eigenvectors $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ are the non-zero solutions of

$$(A - \sqrt{5}I)\vec{v} = \vec{0} \iff -(1+\sqrt{5})x + 2y = 0 \ .$$

Thus choosing for example $x = 1$ you get $y = -\dfrac{1+\sqrt{5}}{2}$.

For the eigenvalue $\lambda = -\sqrt{5}$, the eigenvectors $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ are the non-zero solutions of

$$(A + \sqrt{5}I)\vec{v} = \vec{0} \iff (-1+\sqrt{5})x + 2y = 0 \, .$$

Thus choosing for example $x = 1$ you get $y = \dfrac{-1+\sqrt{5}}{2}$.

7. Consider the function

$$f(x, y) \; = \; xy + y^2 - 3x + 2x^2 - 1.$$

Compute its gradient and its Hessian matrix. Classify its critical points.

Solution:

$$\nabla f \; = \; \begin{pmatrix} y - 3 + 4x \\ x + 2y \end{pmatrix} \quad , \quad \mathscr{H} f \; = \; \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} \, .$$

The critical points of f are the solutions of

$$\nabla f = 0 \iff y - 3 + 4x = 0, \; x + 2y = 0 \iff x = \frac{6}{7}, \; y = -\frac{3}{7} \, .$$

In order to determine the nature of the critical point one can study the sign of the eigenvectors of the Hessian matrix. A real number λ is an eigenvalue o if and only if

$$\det(\mathscr{H} f - \lambda I) = 0 \iff \det \begin{pmatrix} 4-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = 0 \iff \lambda^2 - 6\lambda + 7 = 0 \iff \lambda = 3 \pm \sqrt{2} \, .$$

Therefore the eigenvalues are positive and $\left(\dfrac{6}{7}, -\dfrac{3}{7}\right)$ is a point of minimum for f.

8. Consider the functions

$$f(x, y) \; = \; x + y \quad , \quad g(x, y) = x^2 + y^2 - 1 \, . \tag{1}$$

Use the method of Lagrange Multipliers to find the maximum of f subject to the constraint $g(x, y) = 0$.

Solution:
The Lagrangian is

$$L(x, y) = x + y - \lambda(x^2 + y^2 - 1) \, .$$

Compute the critical points of L:

$$\begin{cases} \nabla L(x, y) = 0 \\ x^2 + y^2 = 1 \end{cases} \iff \begin{cases} 1 - 2\lambda x = 0 \\ 1 - 2\lambda y = 0 \\ x^2 + y^2 = 1 \end{cases} \iff \begin{cases} \lambda = \pm\frac{\sqrt{2}}{2} \\ x = \pm\frac{\sqrt{2}}{2} \\ y = \pm\frac{\sqrt{2}}{2} \end{cases} \, .$$

The Lagrangian has two critical points (corresponding to different values of λ). Since f is continuous on the compact set $\{g = 0\}$, it admits a minimum point and a maximum point. Therefore the critical points of L are the extremal points of f. Since $f(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) > f(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ then the maximum point is $P_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and the minimum point is $P_2 = (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$.

9. Consider the discrete probability space of a dice (with 6 faces) roll.

a) Let us denote by A the event "the result is a divider of 4" and B the event "the result is a prime number". Are A and B independent events? Justify the answer.
b) Compute the expected value and the variance of the random variable $f = (1, 4, -1, 2, 0, 1)$.

Solution:

a) Let us start rewriting A and B as disjoint union of elementary events, i.e.

$$A = \{1\} \cup \{2\} \cup \{4\}$$
$$B = \{1\} \cup \{2\} \cup \{3\} \cup \{5\}.$$

Notice that since the two dice rolls are independent each elementary event has probability $\frac{1}{6}$ and then one can compute

$$P(A) = \frac{1}{2}$$

$$P(B) = \frac{2}{3}$$

$$P(A \cap B) = P(\{1, 2\}) = \frac{1}{3} \quad \text{and} \quad P(A)P(B) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}.$$

Therefore A and B are independent events.

b)

$$\mathbb{E}(f) = \frac{1}{6} + \frac{4}{6} - \frac{1}{6} + \frac{2}{6} + \frac{1}{6} = \frac{7}{6}$$
$$\mathbb{E}(f^2) = \frac{1}{6} + \frac{16}{6} + \frac{1}{6} + \frac{4}{6} + \frac{1}{6} = \frac{23}{6}$$
$$var(f) = \mathbb{E}(f^2) - \mathbb{E}(f)^2 = \frac{23}{6} - \frac{49}{36} = \frac{89}{36}.$$

10. Let X be a random variable with binomial distribution with parameters n=6 number of trials and p=$\dfrac{1}{4}$ probability of success.

a) Compute the probability to have 4 successes.
b) Compute the expected value and the variance of X.
c) Find a value $\delta > 0$ such that

$$\mathbb{P}\left(\left|\frac{X}{6} - \frac{1}{4}\right| \geq \delta\right) \leq 10^{-4};$$

justify your answer.

Solution:

a) $P(X = 4) = \binom{n}{k} p^k (1-p)^{n-k} = \binom{6}{4} \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^2$
b) $\mathbb{E}(X) = np = \frac{3}{2}; \quad var(X) = np(1-p) = \frac{9}{8}.$
c) Recall that the Chebyshev's inequality applied to the binomial distribution gives:

$$\mathbb{P}\left(|\frac{X}{n} - p| \geq \delta\right) \leq \frac{p(1-p)}{n\delta^2}.$$

Therefore you can choose $\delta > 0$ such that

$$\frac{p(1-p)}{n\delta^2} \leq 10^{-4} \iff \frac{\frac{1}{4}\frac{3}{4}}{6\delta^2} \leq 10^{-4} \iff \delta^2 \geq \frac{3}{16 \cdot 6} 10^4 \iff \delta \geq \frac{100}{4\sqrt{2}}.$$